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# On the Ratio between Sector and Triangle in the Orbit of a Celestial Body.

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1. THIS ratio may be expressed by the formula

$$\frac{1}{\eta} = \frac{rr' \sin (v' - v)}{\sqrt{p\tau}} = \frac{\sin 2\theta}{2\theta}, \quad (1)$$

where the mass of the body is neglected,  $r$  and  $r'$  are the radii vectores,  $v$  and  $v'$  the corresponding true anomalies,  $p$  the semi-parameter, and  $\tau$  the product of the intervening time and the constant of the solar system.

By Taylor's theorem, including terms of the fourth order,

$$v = v_0 - \frac{1}{2} \frac{dv_0}{d\tau} \tau + \frac{1}{8} \frac{d^2v_0}{d\tau^2} \tau^2 - \frac{1}{48} \frac{d^3v_0}{d\tau^3} \tau^3 + \frac{1}{384} \frac{d^4v_0}{d\tau^4} \tau^4 - \dots,$$

$$v' = v_0 + \frac{1}{2} \frac{dv_0}{d\tau} \tau + \frac{1}{8} \frac{d^2v_0}{d\tau^2} \tau^2 + \frac{1}{48} \frac{d^3v_0}{d\tau^3} \tau^3 + \frac{1}{384} \frac{d^4v_0}{d\tau^4} \tau^4 + \dots,$$

where  $v_0$  is the true anomaly corresponding to the mean of the times;

$$\therefore v' - v = \frac{dv_0}{d\tau} \tau + \frac{1}{24} \frac{d^3v_0}{d\tau^3} \tau^3 + \dots$$

$$\therefore \sin (v' - v) = \frac{dv_0}{d\tau} \tau + \left( \frac{1}{24} \frac{d^3v_0}{d\tau^3} - \frac{1}{6} \left( \frac{dv_0}{d\tau} \right)^3 \right) \tau^3.$$

The well-known expressions  $\frac{dv_0}{d\tau} = \frac{\sqrt{p}}{r_0^2}$  and  $\frac{p}{r_0} = 1 + e \cos v_0$  give, by successive differentiation,

$$\frac{d^2r_0}{d\tau^2} = \frac{p - r_0}{r_0^3},$$

$$\frac{d^3v_0}{d\tau^3} = \frac{\sqrt{p}}{r_0^2} \left( \frac{6}{r_0^2} \left( \frac{dr_0}{d\tau} \right)^2 - \frac{2}{r_0} \frac{d^2r_0}{d\tau^2} \right),$$

whence by substitution, including terms of the third order,

$$\frac{\sin(v' - v)}{\sqrt{p\tau}} = \frac{1}{r_0^2} \left[ 1 + \left( \frac{1}{4r_0^2} \left( \frac{dr_0}{d\tau} \right)^2 - \frac{3p - r_0}{12r_0^4} \right) \tau^2 + \dots \right]. \quad (2)$$

Developing  $r$  and  $r'$  in the same manner as  $v$  and  $v'$  were developed, and multiplying the results, we have

$$rr' = r_0^2 \left[ 1 - \left( \frac{1}{4r_0^2} \left( \frac{dr_0}{d\tau} \right)^2 - \frac{p - r_0}{4r_0^4} \right) \tau^2 + \dots \right]. \quad (3)$$

The product of (2) and (3) gives

$$\begin{aligned} \frac{1}{\eta} &= 1 - \frac{\tau^2}{6r_0^3} + \dots = 1 - \frac{4\theta^2}{6} + \dots; \\ \therefore 2\theta &= \frac{\tau}{r_0^{\frac{3}{2}}} + \dots = \frac{\tau}{(rr')^{\frac{3}{2}}} + \dots = \frac{2^{\frac{3}{2}}\tau}{(r + r')^{\frac{3}{2}}} + \dots \end{aligned}$$

2. For a closer approximation we have the well-known equations

$$\begin{aligned} \frac{\eta^2}{m} &= \frac{1}{l + \sin^2 \frac{1}{2}g}, \\ \frac{\eta^2}{m}(\eta - 1) &= \frac{2g - \sin 2g}{\sin^3 g}, \end{aligned}$$

in which

$$l = \frac{\sin^2 \frac{1}{2}\gamma}{\cos \gamma}, \quad m = \frac{\theta_0^2}{2 \cos^3 \gamma}, \quad \cos \gamma = \frac{2\sqrt{rr'}}{r + r'} \cos \frac{1}{2}(v' - v), \quad \theta_0^2 = \frac{2\tau^2}{(r + r')^3}$$

and  $g$  is the half difference of the eccentric anomalies.

For the first of these equations we may write

$$x = \sin^2 \frac{1}{2}g = \frac{m}{\eta^2} - l,$$

or by substitution,

$$x = \frac{1}{8} \frac{\theta_0^2}{\theta^2} \frac{\sin^2 2\theta}{\cos^3 \gamma} - \frac{\sin^2 \frac{1}{2}\gamma}{\cos \gamma}.$$

In the same manner the second becomes

$$\frac{\theta^2}{\theta_0^2} \cdot \frac{\cos^3 \gamma}{\cos^3 \theta} \cdot \frac{2\theta - \sin 2\theta}{\sin^3 \theta} = \frac{2g - \sin 2g}{\sin^3 g}.$$

Gauss has developed the second member of the latter equation in a series arranged according to the ascending powers of  $x$ , as follows:

$$\frac{2g - \sin 2g}{\sin^3 g} = \frac{4}{3} \left( 1 + \frac{6}{5}x + \frac{6}{5} \cdot \frac{8}{7}x^2 + \dots \right),$$

whence, if we develop  $\frac{2\theta - \sin 2\theta}{\sin^3 \theta}$  in a similar series arranged according to the ascending powers of  $z = \sin^2 \frac{1}{2} \theta$ , we shall have

$$\theta^2 \cos^3 \gamma \left(1 + \frac{6}{5} z + \dots\right) = \theta_0^2 \cos^3 \theta \left(1 + \frac{6}{5} x + \dots\right),$$

or, dividing by  $\left(1 + \frac{6}{5} z + \dots\right) \left(1 + \frac{6}{5} x + \dots\right)$ ,

$$\theta^2 \cos^3 \gamma \left(1 - \frac{6}{5} x + \dots\right) = \theta_0^2 \cos^3 \theta \left(1 - \frac{6}{5} z + \dots\right).$$

Substituting the values of  $x$  and  $z$ , and reducing,

$$\theta^2 \cos^2 \gamma \left(1 - \frac{4}{5} \sin^2 \frac{1}{2} \gamma + \dots\right) = \theta_0^2 \cos^2 \theta \left(1 - \frac{4}{5} \sin^2 \frac{1}{2} \theta + \dots\right),$$

whence, approximately,

$$\theta = \theta_0 \left(\frac{\cos \theta}{\cos \gamma}\right)^{1.2}.$$

This formula includes terms of the same order as those included in Hansen's method, and, if employed in connection with a table giving the logarithms of the ratios between sines and arcs, is rather more convenient.